Linear Aerodynamic Theory of Rotor Blades

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A theory is developed for predicting the spanwise lift distribution of rotor blades in forward flight. The guiding thought of Reissner's theory for oscillating wings of finite span is followed, that is, the theory starts with the lifting-surface theory and then equations of the lifting-line theory are derived by mathematical approximation. The equation of the lifting-surface theory is exact within the scope of the linear theory. An approximation equivalent to that of Weissinger's L method is adopted rather than that of Prandtl's lifting-line theory. Numerical computations are carried out for the case of an NACA experiment. Fairly good agreements are obtained with the experimental values. However, the theory seems to fail to predict the abrupt lift changes, which are probably due to the nonlinear effect.

Nomenclature

$A(k_p)$ = unsteadiness function, Eq. (74)
$B(k_p)$ = another unsteadiness function, Eq. (78)
b = a measure of semichord, Eq. (71)
$D(P_0, P_m')$ = coefficient for $\delta(P_m')$ in integral representation for $w_n(P_0)$
D/Dt = linearized substantial derivative
D_1, G_1, R_1, S_1, T_1 = approximate functions for D, G, R, S, T (ψ_1 has been replaced by ψ)
E, F, G = first fundamental quantities of reference surface
E_0, F_0, G_0 = values of E, F, G at $\sigma = \tau = 0$
$G(P_0, P_m')$ = coefficient for $\gamma(P_m')$ in integral representation for $w_n(P_0)$
G_p = complex Fourier coefficient of Γ_0
H = $(EG - F^2)^{1/2}$
h = a measure of pitch of spiral surface (reference surface), Eq. (3)
J = sum of terms independent of τ in simplified lifting-surface equation
k_p = reduced frequency for pth harmonic
$\mathbf{k}$ = unit vector in z direction
N = number of blades
n_b = deviation of blade surface from reference surface
$\mathbf{n}$ = unit normal vector of reference surface
P_0, P_m' = points ($\tau, r, \sigma = 0$) and ($\tau', r', \sigma' = \sigma_m$), respectively
p = pressure, also harmonic order of circulation
$R(P_0, P_m')$ = distance between P_0 and P_m'
r_1, r_2 = blade root and tip radii, respectively
$\mathbf{r}_r, \mathbf{r}_\tau$ = tangent vectors to parametric curves on reference surface
$S(P_0, P_0'), T(P_0, P_0')$ = see Eq. (40)
w_n = induced velocity normal to reference surface
Γ = circulation
γ, δ = components of vorticity vector (not necessarily of physical dimension)
Δ = circulation moment
λ, μ = see Eqs. (38)
τ_1, τ_2 = τ coordinates of blade leading and trailing edges
τ_{1m}', τ_{2m}' = τ_1, τ_2 of mth blade at r'
$\tilde{\tau}$ = τ coordinate of blade midchord (approximately), Eq. (71)
χ = angle of uniform flow against z axis
ψ = azimuth angle of reference blade
ψ_1 = $\psi - \tau/2$
Ω = angular velocity of blade rotation

Subscripts

b, w = blade and wake, respectively
m = serial number of blade, reference blade corresponds to $m = 0$
U, L = upper and lower surface, respectively

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Introduction

It is well known that the conventional blade element theory is of little use for the prediction of complicated lift variations of the rotor blade in forward flight. Recently, attempts were made to attack this problem along the lines of the well-established wing theory.

Willmer[1] used a simple quasi-steady model in which a spiral trailing vortex sheet was replaced by a series of rectangular sheets. Miller[2] assumed a wake vortex sheet partly straight and partly spiral from the standpoint of the unsteady wing theory, and considered the effect of the wake distortion in some degree.

The purpose of this paper is to present a more rigorous and systematic treatment within the scope of the linear theory. The development is a natural extension of the unsteady wing theory and follows Reissner's guiding thought[3] to deduce a lifting-line theory from a lifting-surface theory by mathematical approximation.

By introducing an appropriate curvilinear coordinate system, an integral equation of the lifting-surface theory is derived from the standpoint of the vortex theory. This equation is reduced to equations of lifting-line theory through the simplification of the kernels and the chordwise weighted integration. The last step of the simplification of the kernels is equivalent to the approximation in Weissinger's L method[4] rather than that in Prandtl's lifting-line theory, because the flow direction is not generally perpendicular to the blade span. The equations of the lifting-line theory consist of two equations, that is, an equation of the circulation and an equation of the "circulation moment," from the solutions of which the blade section lift can be calculated.

The distortion of the wake vortex sheet due to the induced flow or to the rolling-up phenomenon is neglected in order to retain the linearity of the problem.

Formulation of the Problem

In the linear theory of the lifting surface for nonrotating wings, the tangential flow condition is imposed at the xy plane and the surface of discontinuity is placed on the same plane. This plane may be called a reference plane.

In the linear theory of the rotor, a "reference surface" of each blade is the spiral surface swept by the rotating radius representing the blade. The surface swept by the actual blade lies in the neighborhood of the reference surface. It is convenient to introduce a system of curvilinear coordinates such that one of the coordinates takes a constant value on each reference surface. This will be carried out through the following three steps.

First, let O-xyz be a rectangular coordinate system that is translated but does not rotate with the rotor (Fig. 1). The x axis is taken backward and the z axis downward. The blade is assumed to rotate in the neighborhood of the xy plane.

Secondly, "skew cylindrical coordinates" ζ, r, θ are introduced, the ζ axis being taken in the uniform flow direction. By referring to Fig. 1, relations to the first coordinates are seen to be

$$x = \zeta \sin\chi + r\cos(\psi - \theta) \qquad y = -r\sin(\psi - \theta) \qquad z = \zeta\cos\chi \tag{1}$$

where ψ is the azimuth angle of a reference blade and χ is the angle between the ζ and z axes.

Finally, σ and τ are defined by

$$\sigma = \theta - \zeta/h \qquad \tau = \theta + \zeta/h \tag{2}$$

where

$$h = (U^2 + W^2)^{1/2}/\Omega \tag{3}$$

U and W are the x and z components of the uniform flow velocity. Ω is the angular velocity of the blade rotation. r, σ, τ thus introduced will be used as coordinates throughout.

With these coordinates the reference surfaces are given simply by

$$\sigma = \sigma_m \equiv 2m\pi/N \qquad (m = 0, 1, 2, \ldots, N-1) \tag{4}$$

where N is the number of blades. From Eqs. (1, 2, and 4), the parametric equations of the reference surface can be written as

$$\left.\begin{aligned} x &= f_1(\tau, r, \psi) \equiv \frac{1}{2}h(\tau - \sigma_m)\sin\chi + r\cos\left(\psi - \frac{\tau + \sigma_m}{2}\right) \\ y &= f_2(\tau, r, \psi) \equiv -r\sin\left(\psi - \frac{\tau + \sigma_m}{2}\right) \\ z &= f_3(\tau, r, \psi) \equiv \tfrac{1}{2}h(\tau - \sigma_m)\cos\chi \end{aligned}\right\} \tag{5}$$

A position vector of a point on the reference surface is given by

$$\mathbf{r}(\tau, r, \psi) = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k} \tag{6}$$

where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are unit vectors in the x, y, and z directions. Write

$$\partial\mathbf{r}/\partial\tau = \mathbf{r}_\tau \qquad \partial\mathbf{r}/\partial r = \mathbf{r}_r \tag{7}$$

Then $\mathbf{r}_\tau$ and $\mathbf{r}_r$ represent, respectively, tangent vectors to the parametric curves given by r = constant and τ = constant on the reference surface. The first fundamental quantities of the reference surface are given by

$$\left.\begin{aligned} E &= \mathbf{r}_\tau\cdot\mathbf{r}_\tau = \frac{1}{4}\left[h^2 + r^2 + 2hr\sin\chi \times \sin\left(\psi - \frac{\tau + \sigma_m}{2}\right)\right] \\ F &= \mathbf{r}_\tau\cdot\mathbf{r}_r = \frac{1}{2}h\sin\chi\cos\left(\psi - \frac{\tau + \sigma_m}{2}\right) \\ G &= \mathbf{r}_r\cdot\mathbf{r}_r = 1 \end{aligned}\right\} \tag{8}$$

A unit normal vector of the reference surface is

$$\mathbf{n} = (\mathbf{r}_r \times \mathbf{r}_\tau)/H \tag{9}$$

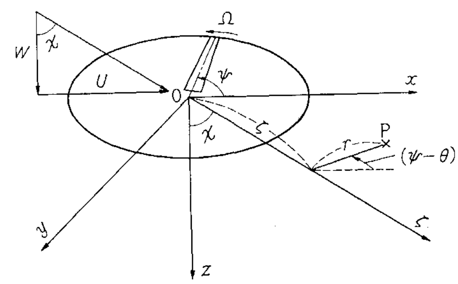

Fig. 1 Coordinate systems.

where

$$H = (EG - F^2)^{1/2} = \frac{1}{2}\left\{(h\cos\chi)^2 + \left[r + h\sin\chi \times \sin\left(\psi - \frac{\tau + \sigma_m}{2}\right)\right]^2\right\}^{1/2} \tag{10}$$

In the following, incompressible flow and small disturbance are assumed. The blade may be represented by a surface of discontinuity of pressure and tangential velocity, whereas in the wake a surface of discontinuity of tangential velocity may be formed. By virtue of the assumption of small disturbance, it may be possible to place both these surfaces of discontinuity on the reference surface. Outside the surfaces of discontinuity, there exists a disturbance potential ϕ in terms of which a disturbance velocity vector is given by

$$\mathbf{q} = \text{grad}\phi \tag{11}$$

Let p designate the pressure and ρ the density; then a linearized form of Kelvin's equation is

$$(p_\infty - p)/\rho = D\phi/Dt \tag{12}$$

where

$$\frac{D(\ )}{Dt} = \frac{\partial(\ )}{\partial t} + U\frac{\partial(\ )}{\partial x} + W\frac{\partial(\ )}{\partial z} = \Omega\left[\frac{\partial(\ )}{\partial\psi} + 2\frac{\partial(\ )}{\partial\tau}\right] \tag{13}$$

represents a linearized substantial derivative.

Let a side of the surface of discontinuity on which the unit normal vector given by Eq. (9) stands be called an upper surface and another side a lower surface. Also, let quantities on two sides be distinguished by subscripts U and L. Then

$$\boldsymbol{\omega} = (\mathbf{q}_U - \mathbf{q}_L) \times \mathbf{n} \tag{14}$$

gives a vorticity vector of the surface of discontinuity or the vortex sheet. By using Eqs. (9) and (11), and by writing

$$\gamma = \left(\frac{\partial\phi}{\partial\tau}\right)_U - \left(\frac{\partial\phi}{\partial\tau}\right)_L \qquad \delta = \left(\frac{\partial\phi}{\partial r}\right)_U - \left(\frac{\partial\phi}{\partial r}\right)_L \tag{15}$$

Eq. (14) becomes

$$\boldsymbol{\omega} = (\gamma\mathbf{r}_r - \delta\mathbf{r}_\tau)/H \tag{16}$$

Now, introduce a vector potential

$$\boldsymbol{\psi}(P) = \frac{1}{4\pi}\Sigma\iint\frac{\boldsymbol{\omega}(P')}{R(P, P')}dS(P') \tag{17}$$

Then the velocity induced by the vortex sheet is given by[5]

$$\mathbf{q} = \text{curl}\boldsymbol{\psi} \tag{18}$$

In Eq. (17), a point (τ, r, σ) has been designated by P and a point (τ', r', σ') by P'. $R(P, P')$ represents a distance between P and P'. The range of the surface integral is the whole vortex sheet of one blade, and Σ means a summation with respect to all the blades.

If P is specified to be a point on one of the reference surfaces, a component of $\mathbf{q}$ in the direction of $\mathbf{n}$ at point P, w_n, is given by

$$w_n(P) = \mathbf{n}(P)\cdot\mathbf{q}(P) = \frac{1}{4\pi}\Sigma\iint \boldsymbol{\omega}(P')\cdot\left[\mathbf{n}(P)\times\text{grad}\left(\frac{1}{R}\right)\right]dS(P')$$

By substituting Eqs. (9) and (16) and by transforming the surface integral to a successive integral with respect to τ' and r', there follows

$$w_n(P) = \frac{1}{4\pi H(P)}\times \Sigma\iint\frac{G(P, P')\gamma(P') + D(P, P')\delta(P')}{[R(P, P')]^3}d\tau'dr' \quad (19)$$

where

$$\left.\begin{aligned}\frac{G}{R^3} &= [\mathbf{r}_\tau(P)\cdot\mathbf{r}_\tau(P')]\frac{\partial(1/R)}{\partial r} - [\mathbf{r}_r(P)\cdot\mathbf{r}_\tau(P')]\frac{\partial(1/R)}{\partial\tau}\\ \frac{D}{R^3} &= [\mathbf{r}_r(P)\cdot\mathbf{r}_r(P')]\frac{\partial(1/R)}{\partial\tau} - [\mathbf{r}_\tau(P)\cdot\mathbf{r}_r(P')]\frac{\partial(1/R)}{\partial r}\end{aligned}\right\} \quad (20)$$

From Eq. (15) it follows that γ and δ are related to each other in the form

$$\partial\gamma/\partial r = \partial\delta/\partial\tau \quad (21)$$

From combination of Eqs. (12) and (15), it follows that the discontinuity of pressure $\Delta p = p_U - p_L$ is given by

$$-\frac{\Delta p}{\rho\Omega} = \frac{\partial}{\partial\psi}\int_{-\infty}^{\tau}\gamma d\tau + 2\gamma \quad (22)$$

The problem of rotor blades as a system of rotating lifting surfaces may now be formulated. Designate the blade root and tip radii by r_1 and r_2, and let the equations of the leading and trailing edges be $\tau = \tau_1(r, \psi)$ and $\tau = \tau_2(r, \psi)$, respectively. The following three regions can be distinguished in the reference surface of each blade: 1) the blade region R_b, 2) the wake region $R_w(\tau_2 < \tau < \infty$ and $r_1 < r < r_2)$, 3) the remaining region R_r.

There is the condition of continuous flow in R_r,

$$\gamma = \delta = 0 \quad (23)$$

and the condition of continuous pressure in R_w,

$$\Delta p = 0 \quad (24)$$

In the blade region R_b, the condition of tangential flow is to be satisfied. If $n_b(\tau, r, \psi)$ stands for the deviation of the blade surface from the reference surface at a point (τ, r, ψ) on the latter measured along the normal $\mathbf{n}$, this condition is expressed by

$$w_n(\tau, r, \psi) = Dn_b/Dt \quad (25)$$

At the trailing edge of the blade, the Kutta condition is imposed, stating that γ and δ are finite there.

In addition, in the following, it is assumed that the blade motion is a function only of the azimuth angle and that this function is the same for all the blades. Therefore, γ and δ at a certain point on one blade are also functions only of the azimuth angle, and can be related to those at the corresponding point of the reference blade by

$$\begin{aligned}\gamma(\tau, r, \sigma_m, \psi) &= \gamma(\tau - 2m\pi/N, r, 0, \psi - 2m\pi/N)\\ \delta(\tau, r, \sigma_m, \psi) &= \delta(\tau - 2m\pi/N, r, 0, \psi - 2m\pi/N)\end{aligned} \quad (26)$$

Equations (26) hold also for γ and δ in the wake region.

By introducing Eqs. (23–26) in the integral representation for w_n, Eq. (19), and by taking account of Eqs. (21) and (22), the general integral equation of lifting-surface theory for the distribution of γ and δ over the blade region of the reference blade may be obtained. With the solution of this integral equation, Eq. (22) furnishes the pressure distribution over the blade.

Designate γ and δ in R_b by γ_b and δ_b and those in R_w by γ_w and δ_w. Let the circulation Γ be defined by

$$\Gamma = \int_{\tau_1}^{\tau_2}\gamma_b\,d\tau \quad (27)$$

Then in the wake region R_w, by means of Eqs. (23, 24, and 27), Eq. (22) takes on the form

$$2\gamma_w + \frac{\partial}{\partial\psi}\int_{\tau_2}^{\tau}\gamma_w d\tau = -\frac{\partial\Gamma}{\partial\psi} \quad (28)$$

The solution of Eq. (28) is

$$\begin{aligned}\gamma_w(\tau, r, \psi) &= -\tfrac{1}{2}[\partial\Gamma(r, \psi')/\partial\psi']_{\psi'=\psi-(\tau-\tau_2)/2}\\ &= \frac{\partial}{\partial\tau}\Gamma\left(r, \psi - \frac{\tau-\tau_2}{2}\right)\end{aligned} \quad (29)$$

When γ_w of Eq. (29) is substituted in Eq. (21), there is obtained for δ_w,

$$\delta_w(\tau, r, \psi) = \int_{-\infty}^{\tau}\frac{\partial\gamma}{\partial r}d\tau' = \frac{\partial}{\partial r}\Gamma\left(r, \psi - \frac{\tau-\tau_2}{2}\right) \quad (30)$$

Similarly,

$$\delta_b = \int_{\tau_1}^{\tau}\frac{\partial\gamma_b}{\partial r}d\tau' \quad (31)$$

By applying Eqs. (25–27) and (29–31) to Eq. (19), a lifting-surface equation whose unknown function is γ_b of the reference blade will be obtained. However, the main purpose of this paper is to develop a lifting-line theory, an approximate theory that is valid when the aspect ratio is large. The lifting-surface equation serves only as the first step to the purpose, and so its full expression will not be given.

Simplification of Lifting-Surface Equation

The second step to the lifting-line theory is to simplify and to modify the lifting-surface equation on the assumption that the blade aspect ratio is sufficiently large.

Divide the right-hand side of Eq. (19) into five parts as

$$w_n(P_0) = I_1 + I_2 + I_3 + I_4 + I_5 \quad (32)$$

where

$$\left.\begin{aligned}I_1 &= \frac{1}{4\pi H(P_0)}\int_{r_1}^{r_2}\int_{\tau_{10}'}^{\infty}\frac{G(P_0, P_0')\gamma(P_0')}{[R(P_0, P_0')]^3}d\tau'dr'\\ I_2 &= \frac{1}{4\pi H(P_0)}\int_{r_1}^{r_2}\int_{\tau_{10}'}^{\infty}\frac{D(P_0, P_0')\delta(P_0')}{[R(P_0, P_0')]^3}d\tau'dr'\\ I_3 &= \frac{1}{4\pi H(P_0)}\sum_{m=1}^{N-1}\int_{r_1}^{r_2}\int_{\tau_{1m}'}^{\tau_{2m}'}\times\\ &\qquad\frac{G(P_0, P_m')\gamma_b(P_m') + D(P_0, P_m')\delta_b(P_m')}{[R(P_0, P_m')]^3}d\tau'dr'\\ I_4 &= \frac{1}{4\pi H(P_0)}\times\\ &\qquad\sum_{m=1}^{N-1}\int_{r_1}^{r_2}\int_{\tau_{2m}'}^{\infty}\frac{G(P_0, P_m')\gamma_w(P_m')}{[R(P_0, P_m')]^3}d\tau'dr'\\ I_5 &= \frac{1}{4\pi H(P_0)}\times\\ &\qquad\sum_{m=1}^{N-1}\int_{r_1}^{r_2}\int_{\tau_{2m}'}^{\infty}\frac{D(P_0, P_m')\delta_w(P_m')}{[R(P_0, P_m')]^3}d\tau'dr'\end{aligned}\right\} \quad (33)$$

P_0 and P_m' represent points ($\tau, r, \sigma = 0$) and ($\tau', r', \sigma' = \sigma_m$) ($m = 0, 1, 2, \ldots, N-1$), respectively. τ_{1m}' and τ_{2m}' are the τ coordinates of the leading and trailing edges of the mth blade, primes indicating that they are values at a radius r'.

The distance between P_0 and P_m' is given by

$$R(P_0, P_m') = \{h^2\lambda^2 + r^2 + r'^2 - 2rr' \cos\mu - 2(h \sin\chi)\lambda[r \cos\psi_1 - r' \cos(\psi_1 - \mu)]\}^{1/2} \quad (34)$$

Full expressions of $G(P_0, P_m')$ and $D(P_0, P_m')$ are

$$G(P_0, P_m') = -\tfrac{1}{2}\{r^2 \sin\mu + h^2\lambda \cos\mu + (h \sin\chi)r[\sin\psi_1 \sin\mu + \lambda \sin(\psi_1 - \mu)] - (h \sin\chi)^2\lambda \cos\psi_1 \cos(\psi_1 - \mu)\} \quad (35)$$

$$D(P_0, P_m') = \tfrac{1}{4}\{r(h^2 - r'^2) + (r^2 - h^2)r' \cos\mu - h^2r'\lambda \sin\mu + (h \sin\chi)[(r^2 - r'^2) \sin\psi_1 + rr' \cos\psi_1 \sin\mu - rr'\lambda \cos(\psi_1 - \mu)] + (h \sin\chi)^2 \times (\cos\psi_1)[-r \cos\psi_1 + r' \cos(\psi_1 - \mu) - r'\lambda \sin(\psi_1 - \mu)]\} \quad (36)$$

where

$$\psi_1 = \psi - \tau/2 \quad (37)$$

$$\lambda = (\tau' - \tau)/2 - m\pi/N \qquad \mu = (\tau' - \tau)/2 + m\pi/N \quad (38)$$

It can be observed from Eq. (35) that $G(P_0, P_m')$ is independent of r'.

Simplification of I_1

After integrating I_1 by parts with respect to r', divide it into two parts as follows:

$$I_1 = I_1^{(1)} + I_1^{(2)} \quad (39)$$

$$\left.\begin{aligned} I_1^{(1)} &= -\frac{1}{4\pi H}\int_{r_1}^{r_2}\int_{\tau_{10}'}^{\tau_{20}'} \frac{TG}{SR}\frac{\partial\gamma_{b0}}{\partial r'}\, d\tau' dr' \\ I_1^{(2)} &= -\frac{1}{4\pi H}\int_{r_1}^{r_2}\int_{\tau_{10}'}^{\tau_{20}'} \frac{TG}{SR}\frac{\partial\gamma_{w0}}{\partial r'}\, d\tau' dr' \end{aligned}\right\} \quad (40)$$

where γ_{b0} and γ_{w0} mean $\gamma_b(P_0')$ and $\gamma_w(P_0')$. Both S and T are functions of P_0 and P_0', and are given by

$$S(P_0, P_0') = (h \cos\chi)^2\lambda^2 + [r \sin\lambda + (h \sin\chi)\lambda \sin(\psi_1 - \lambda)]^2 \quad (41)$$

$$T(P_0, P_0') = r' - r \cos\lambda + (h \sin\chi)\lambda \cos(\psi_1 - \lambda) \quad (42)$$

where in this case λ denotes $(\tau' - \tau)/2$ corresponding to $m = 0$.

First, since $I_1^{(1)}$ is an integral over the blade region of the reference blade, $|\tau' - \tau|$ and $|\tau|$ may be considered small there from the assumption that the aspect ratio is large. Then the following approximation may be possible[6]:

$$I_1^{(1)} \approx \frac{1}{4\pi H_0}\int_{r_1}^{r_2}\int_{\tau_{10}'}^{\tau_{20}'} \frac{F_0(\tau' - \tau) + (r' - r)}{(\tau' - \tau)[E_0(\tau' - \tau)^2 + 2F_0(\tau' - \tau)(r' - r) + (r' - r)^2]^{1/2}} \times \frac{\partial\gamma_{b_0}}{\partial r'}\, d\tau' dr' \quad (43)$$

where

$$E_0 = E(P_0)|_{\tau=0} \qquad F_0 = F(P_0)|_{\tau=0} \qquad H_0 = H(P_0)|_{\tau=0} \quad (44)$$

Next, in $I_1^{(2)}$, $|\tau|$ may be considered small so that functions R, G, S, and T are replaced by R_1, G_1, S_1, and T_1, in which $\psi_1 = \psi - \tau/2$ has been approximated by ψ. If the circulation of the reference blade Γ_0 is expanded in a complex Fourier series

$$\Gamma_0(r', \psi) = \sum_{p=-\infty}^{\infty} G_p(r')e^{ip\psi} \quad (45)$$

then by using Eq. (29) and considering that τ_{20}' may also be small, γ_{w_0} is given by

$$\gamma_{w_0}(\tau', r', \psi) \approx -\left(\frac{i}{2}\right)\sum_{p=-\infty}^{\infty} pG_p(r')e^{ip(\psi-\lambda)} \quad (46)$$

Now, introduce an integral over a fictious wake

$$\begin{aligned} I_1^{(3)} &= -\frac{1}{2\pi H_0}\int_{\tau_{20}(r)}^{\infty} \frac{\gamma_{w_0}(r)}{\tau' - \tau}\, d\tau' \\ &= \frac{1}{4\pi H_0}\int_{r_1}^{r_2}\int_{\tau_{20}(r)}^{\infty} \frac{r' - r}{|r' - r|}\frac{1}{\tau' - \tau}\frac{\partial\gamma_{w_0}}{\partial r'}\, d\tau' dr' \end{aligned} \quad (47)$$

where $\gamma_{w0}(r)$ and $\tau_{20}(r)$ represent values at radius r (not at r'). Rewrite Eq. (39) as follows:

$$I_1 = I_1^{(1)} + I_1^{(3)} + (I_1^{(2)} - I_1^{(3)}) \quad (48)$$

In the last term $(I_1^{(2)} - I_1^{(3)})$, the two integrands almost cancel each other when $|\tau'|$ is small so that the lower ends of the τ' intervals of the both integrals may be replaced by τ. Furthermore, by transforming the integration variable τ' into λ and by using Eq. (46), $(I_1^{(2)} - I_1^{(3)})$ is approximated by

$$I_1^{(2)} - I_1^{(3)} \approx \frac{i}{4\pi H_0} \times \sum_{p=-\infty}^{\infty} pe^{ip\psi}\int_{r_1}^{r_2}\frac{dG_p}{dr'}\int_0^{\infty}\left(\frac{T_1G_1}{S_1R_1} + \frac{r' - r}{|r' - r|}\frac{1}{2\lambda}\right)e^{-ip\lambda}\, d\lambda dr' \quad (49)$$

which is independent of τ. $I_1^{(3)}$, the second term of the right-hand side of Eq. (48), is written in the form

$$I_1^{(3)} = \frac{i}{4\pi H_0}\sum_{p=-\infty}^{\infty} pG_p(r)e^{ip[\psi + \tau_{20}(r)/2]}\int_{\tau_{20}(r)}^{\infty}\frac{e^{-ip\tau'/2}}{\tau' - \tau}\, d\tau' \quad (50)$$

Simplification of I_2

Also divide I_2 into two parts as follows:

$$I_2 = I_2^{(1)} + I_2^{(2)} \quad (51)$$

$$\left.\begin{aligned} I_2^{(1)} &= \frac{1}{4\pi H}\int_{r_1}^{r_2}\int_{\tau_{10}'}^{\tau_{20}'} \frac{D\delta_{b_0}}{R^3}\, d\tau' dr' \\ I_2^{(2)} &= \frac{1}{4\pi H}\int_{r_1}^{r_2}\int_{\tau_{20}'}^{\infty} \frac{D\delta_{w_0}}{R^3}\, d\tau' dr' \end{aligned}\right\} \quad (52)$$

Since $I_2^{(1)}$ is an integral over the reference blade, by approximations[6] similar to those for $I_1^{(1)}$

$$I_2^{(1)} \approx -\frac{H_0}{4\pi}\int_{r_1}^{r_2}(r' - r) \times \int_{\tau_{10}'}^{\tau_{20}'}\frac{\delta_{b_0}}{[E_0(\tau' - \tau)^2 + 2F_0(\tau' - \tau)(r' - r) + (r' - r)^2]^{3/2}} \times d\tau' dr' \quad (53)$$

By using Eq. (31) and by assuming that $\partial\tau_{20}'/\partial r'$ is small

$$\delta_{b_0}|_{\tau'=\tau_{20}'} \approx \partial\Gamma_0(r')/\partial r \quad (54)$$

By integrating Eq. (53) by parts with respect to τ'. and by using Eqs (54) and (21), there follows

$$I_2^{(1)} = -\frac{1}{4\pi H_0}\int_{r_1}^{r_2} \frac{E_0(\tau_{20}-\tau)+F_0(r'-r)}{(r'-r)[E_0(\tau_{20}-\tau)^2+2F_0(\tau_{20}-\tau)(r'-r)+(r'-r)^2]^{1/2}}\frac{\partial\Gamma_0}{\partial r'}dr' + \frac{1}{4\pi H_0}\int_{r_1}^{r_2}\int_{\tau_{10}'}^{\tau_{20}'} \frac{E_0(\tau'-\tau)+F_0(r'-r)}{(r'-r)[E_0(\tau'-\tau)^2+2F_0(\tau'-\tau)(r'-r)+(r'-r)^2]^{1/2}}\frac{\partial\gamma_{b_0}}{\partial r'}d\tau'dr' \quad (55)$$

As for $I_2^{(2)}$, functions R and D are replaced by R_1 and D_1, in which $\psi_1 = \psi - \tau/2$ has been approximated by ψ as in the case of $I_1^{(2)}$. By applying Eq. (45) to Eq. (30), δ_{w_0} is

$$\delta_{w_0}(\tau', r', \psi) \approx \sum_{p=-\infty}^{\infty} (dG_p/dr')e^{ip(\psi-\lambda)} \quad (56)$$

Now, introduce an integral over another fictious wake

$$I_2^{(3)} = -\frac{H_0}{4\pi}\int_{r_1}^{r_2}(r'-r)\frac{\partial\Gamma_0}{\partial r'}\int_{\tau_{20}'}^{\infty}\frac{1}{[E_0(\tau'-\tau)^2+2F_0(\tau'-\tau)(r'-r)+(r'-r)^2]^{3/2}}d\tau'dr' = -\frac{1}{4\pi H_0}\int_{r_0}^{r_2}\left\{E_0^{1/2} - \frac{E_0(\tau_{20}'-\tau)+F_0(r'-r)}{[E_0(\tau_{20}'-\tau)^2+2F_0(\tau_{20}'-\tau)(r'-r)+(r'-r)^2]^{1/2}}\right\}\frac{1}{r'-r}\frac{\partial\Gamma_0}{\partial r'}dr' \quad (57)$$

Rewrite Eq. (51) as follows:

$$I_2 = (I_2^{(1)} + I_2^{(3)}) + (I_2^{(2)} - I_2^{(3)}) \quad (58)$$

From combination of Eqs. (55) and (57) there follows

$$I_2^{(1)} + I_2^{(3)} = -\frac{E_0^{1/2}}{4\pi H_0}\int_{r_1}^{r_2}\frac{1}{r'-r}\frac{\partial\Gamma_0}{\partial r'}dr' + \frac{1}{4\pi H_0}\times \int_{r_1}^{r_2}\int_{\tau_{10}'}^{\tau_{20}'}\frac{E_0(\tau'-\tau)+F_0(r'-r)}{(r'-r)[E_0(\tau'-\tau)^2+2F_0(\tau'-\tau)(r'-r)+(r'-r)^2]^{1/2}}\frac{\partial\gamma_{b0}}{\partial r'}d\tau'dr' \quad (59)$$

In the difference $(I_2^{(2)} - I_2^{(3)})$, the interval of integration with respect to τ' is extended to τ as before, and the integration variable is transformed into λ so that

$$I_2^{(2)} - I_2^{(3)} \approx \frac{1}{4\pi H_0}\sum_{p=-\infty}^{\infty} e^{ip\psi}\int_{r_1}^{r_2}\frac{dG_p}{dr'}\left\{2\int_0^{\infty}\frac{D_1}{R_1^3}e^{ip\lambda}d\lambda + \frac{E_0^{1/2}}{r'-r} - \frac{F_0}{|r'-r|}\right\}dr' \quad (60)$$

which does not depend on τ.

Simplification of I_3

Although I_1 and I_2 in the foregoing represent contributions of the vortices of the reference blade, I_3 represents the total contribution of the vortices over the blade regions of all the other blades. Here, a model is considered in which γ_b's concentrate on their respective blade axes, that is, lines $\sigma = \tau = 2m\pi/N$ $(m = 1, 2, \ldots, N-1)$. The contribution of δ_b's may be considered small and is neglected entirely.

By virtue of these two assumptions and by taking into account the fact that $|\tau|$ is small, it may be possible to replace τ by zero and τ' by $2m\pi/N$ in the functions $R(P_0, P_m')$ and $G(P_0, P_m')$ in I_3. With these replacements, the integration by τ' yields a circulation of the mth blade

$$\Gamma_m(r', \psi) = \sum_{p=-\infty}^{\infty} G_p(r')e^{ip(\psi-2m\pi/N)} \quad (61)$$

As a result, simplified I_3 is

$$I_3 \approx -\frac{r(r+h\sin\chi\sin\psi)}{8\pi H_0}\sum_{p=-\infty}^{\infty}e^{ip\psi}\sum_{m=1}^{N-1}e^{-ip(2m\pi/N)}\sin\left(\frac{2m\pi}{N}\right)\int_{r_1}^{r_2}\frac{G_p(r')}{[r^2+r'^2-2rr'\cos(2m\pi/N)]^{3/2}}dr' \quad (62)$$

This also is independent of τ.

Simplification of I_4 and I_5

I_4 and I_5 are contributions of the vortices of the wake regions of all the blades other than the reference blade. As before, replace the functions R, G, and D by R_1, G_1, and D_1, in which ψ_1 has been approximated by ψ. By the approximation

$$\tau_{2m}' \approx \tau + 2m\pi/N \quad (63)$$

and from combination of Eqs. (29, 30, 61, and 38), γ_{wm} and δ_{wm} are

$$\gamma_{wm}(\tau', r', \psi) \approx -\left(\frac{i}{2}\right)\sum_{p=-\infty}^{\infty} pG_p(r')e^{ip(\psi-\mu)} \quad (64)$$

$$\delta_{wm}(\tau', r', \psi) \approx \sum_{p=-\infty}^{\infty}\left(\frac{dG_p}{dr'}\right)e^{ip(\psi-\mu)} \quad (65)$$

By using Eqs. (64) and (65), transforming the variable of integration, and taking into account the fact that $|\tau|$ is small, there follows

$$I_4 \approx -\frac{i}{4\pi H_0}\sum_{p=-\infty}^{\infty} pe^{ip\psi}\int_{r_1}^{r_2} G_p(r')\sum_{m=1}^{N-1}e^{-ip(2m\pi/N)}\int_0^{\infty}\frac{G_1}{R_1^3}e^{-ip\lambda}d\lambda dr' \quad (66)$$

$$I_5 \approx \frac{1}{2\pi H_0}\sum_{p=-\infty}^{\infty} e^{ip\psi}\int_{r_1}^{r_2}\frac{dG_p}{dr'}\sum_{m=1}^{N-1}e^{-ip(2m\pi/N)}\int_0^{\infty}\frac{D_1}{R_1^3}e^{-ip\lambda}d\lambda dr' \quad (67)$$

These are once again independent of τ.

Simplified Lifting-Surface Equation

If the previously obtained simplified integrals are summed up, there results the simplified lifting-surface equation

$$w_n(P_0) = \frac{1}{4\pi H_0}\int_{r_1}^{r_2}\int_{\tau_{10}'}^{\tau_{20}'} \frac{[E_0(\tau'-\tau)^2 + 2F_0(\tau'-\tau)(r'-r) + (r'-r)^2]^{1/2}}{(\tau'-\tau)(r'-r)} \frac{\partial \gamma_{b_0}}{\partial r'} d\tau' dr' + \frac{i}{4\pi H_0}\sum_{p=-\infty}^{\infty} pG_p(r)\, e^{ip[\psi - \tau_{20}(r)/2]} \int_{\tau_{20}(r)}^{\infty} \frac{e^{-ip\tau'/2}}{\tau'-\tau} d\tau' + J \quad (68)$$

in which $I_1^{(1)}$ and the second term of $(I_2^{(1)} + I_2^{(3)})$ have been combined into a single integral. J is the sum of terms independent of τ, and is given by

$$J = -\frac{E_0^{1/2}}{4\pi H_0}\int_{r_1}^{r_2} \frac{1}{r'-r}\frac{\partial \Gamma_0}{\partial r'} dr' - \frac{r(r + h\sin\chi\sin\psi)}{8\pi H_0} \times \sum_{p=-\infty}^{\infty} e^{ip\psi} \sum_{m=1}^{N-1} e^{-ip(2m\pi/N)} \sin\left(\frac{2m\pi}{N}\right) \times \int_{r_1}^{r_2} \frac{G_p(r')}{[r^2 + r'^2 - 2rr'\cos(2m\pi/N)]^{3/2}} dr' + \frac{i}{4\pi H_0}\sum_{p=-\infty}^{\infty} pe^{ip\psi}\int_{r_1}^{r_2}\frac{dG_p}{dr'}\int_0^{\infty}\left(\frac{T_1}{S_1}\frac{G_1}{R_1}\bigg|_{m=0} + \frac{r'-r}{|r'-r|}\frac{1}{2\lambda}\right) e^{-ip\lambda} d\lambda dr' - \frac{i}{4\pi H_0}\sum_{p=-\infty}^{\infty} pe^{ip\psi} \times \int_{r_1}^{r_2} G_p(r')\sum_{m=1}^{N-1} e^{-ip(2m\pi/N)}\int_0^{\infty}\frac{G_1}{R_1^3} e^{-ip\lambda} d\lambda dr' + \frac{1}{4\pi H_0}\sum_{p=-\infty}^{\infty} e^{ip\psi}\int_{r_1}^{r_2}\frac{dG_p}{dr'}\left\{\frac{E_0^{1/2}}{r'-r} - \frac{F_0}{|r'-r|} + 2\sum_{m=0}^{N-1} e^{-ip(2m\pi/N)}\int_0^{\infty}\frac{D_1}{R_1^3} e^{-ip\lambda} d\lambda\right\} dr' \quad (69)$$

Equations of Lifting-Line Theory

It will be sufficient for theories of propellers to rest on approximations of the same degree as those in the Prandtl lifting-line theory for the wing. However, it is impossible for theories of rotors to be based on such approximations since the flow direction is not necessarily perpendicular to the blade span. Therefore the present theory relies on approximations of the same degree as those in the Weissinger L method,[4] which can be applied also to the swept wing.

Since, in the theory of the steady wing, the section lift is determined by the circulation, the equation of the lifting line theory is no more than an equation of the circulation. In order to determine the section lift of the rotor blade, however, it is necessary, as will be stated later, to know not only the circulation but also another quantity which may be called a "circulation moment" owing to the unsteadiness of the flow. The equations of the lifting-line theory of the rotor blade, therefore, consist of two equations, that is, the equation of the circulation and the equation of the circulation moment.

It may be reasonable to consider that w_n defined by the tangential flow condition is constant along the blade chord corresponding to the lifting-line approximations. In what follows, w_n will be assumed to be independent of τ.

Equation of the Circulation

Define new variables ξ and ξ'

$$\xi = (\tau - \bar{\tau})/b \qquad \xi' = (\tau' - \bar{\tau})/b \quad (70)$$

where b and $\bar{\tau}$ are

$$b = \frac{[\tau_{20}(r) - \tau_{10}(r)]}{2} \qquad \bar{\tau} = \frac{[\tau_{10}(r) + \tau_{20}(r)]}{2} \quad (71)$$

In order to derive the equation of the circulation, first introduce an approximate substitution

$$\tau' - \tau \approx -b \quad (72)$$

into the quadratic form within the brackets in the first integral of Eq. (68). Second, multiply both the sides of Eq. (68) by a weighting function $(1/\pi)[(1+\xi)/(1-\xi)]^{1/2}$, integrate them with respect to ξ from -1 to 1, and in addition use some approximations.[6] Then the equation of the circulation follows:

$$w_n(r, \psi) = -\frac{1}{4\pi H_0 b} \times \int_{r_1}^{r_2} \frac{[E_0 b^2 - 2F_0 b(r'-r) + (r'-r)^2]^{1/2}}{r'-r}\frac{\partial \Gamma_0}{\partial r'} dr' + \frac{1}{2\pi H_0 b}\sum_{p=-\infty}^{\infty} A(k_p)G_p(r)e^{ip\psi} + J \quad (73)$$

where

$$A(k_p) = \begin{cases} (\pi/2)k_p e^{ik_p}[H_0^{(2)}(k_p) - iH_1^{(2)}(k_p)] - 1 & (k_p > 0) \\ 0 & (k_p = 0) \\ -(\pi/2)k_p e^{ik_p}[H_0^{(1)}(-k_p) + iH_1^{(1)}(-k_p)] - 1 & (k_p < 0) \end{cases} \quad (74)$$

$H_0^{(1)}$, $H_0^{(2)}$, $H_1^{(1)}$, and $H_1^{(2)}$ represent Hankel functions. k_p is defined by

$$k_p = pb/2 \quad (75)$$

and is interpreted as the reduced frequency for the pth harmonic. It is possible to verify that Eq. (72) corresponds to the approximation in the Weissinger L method.[6]

Equation of the Circulation Moment

Apply the same process to Eq. (68) as deriving the equation of the circulation except that a weighting function for this case is $(2/\pi)(1-\xi^2)^{1/2}$, and take a difference between the resulting equation and Eq. (73). By again using some approximations,[6] another equation of the lifting-line theory follows:

$$\frac{1}{2b}\int_{r_1}^{r_2}\frac{[E_0 b^2 - 2F_0 b(r'-r) + (r'-r)^2]^{1/2}}{r'-r} \times \left[\frac{\partial \Delta}{\partial r'} + \left(\frac{b}{2} - \bar{\tau}\right)\frac{\partial \Gamma_0}{\partial r'}\right] dr' + \sum_{p=-\infty}^{\infty} B(k_p)G_p(r)e^{ip\psi} = 0 \quad (76)$$

where

$$\Delta(r', \psi) = \int_{\tau_{10}'}^{\tau_{20}'} \tau' \gamma_{b_0}\, d\tau' \quad (77)$$

may be called a circulation moment. $B(k_p)$ is given by

$$B(k_p) = \begin{cases} 1 + 1/ik_p + (\pi/2)H_1^{(2)}(k_p)e^{ik_p} - (\frac{1}{2})A(k_p) & (k_p > 0) \\ 0 & (k_p = 0) \\ 1 + 1/ik_p + (\pi/2)H_1^{(1)}(-k_p)e^{ik_p} - (\frac{1}{2})A(k_p) & (k_p < 0) \end{cases} \quad (78)$$

When the circulation Γ_0 is known, the circulation moment Δ will be obtained from Eq. (76).

The value of b in the equations of the circulation and of the circulation moment is obtained with sufficient accuracy by[6]

$$b(r, \psi) = c(r)/2H_0 \quad (79)$$

where c is the chord length. As for $\bar{\tau}$, it is convenient and may be accepted in ordinary rotor blades to assume that

$$\bar{\tau} = 0 \quad (80)$$

w_n and Section Lift

The value of w_n, which is independent of τ as previously assumed, is obtained by evaluating Eq. (25) at $\tau = 0$. This corresponds roughly to neglecting terms associated with the blade pitching velocity. Equations for this purpose must hold only in the vicinity of the axis of the reference blade:

$$\sigma = \tau = 0 \quad (81)$$

Define a new rectangular right-handed coordinate system O-$\xi\eta z$ rotating with the blade in which the ξ axis coincides with the line given by Eq. (81). In the region where σ and τ are small, there are obtained the relations

$$\left.\begin{aligned} \xi &\approx r + (\tfrac{1}{2})(\tau - \sigma)h \sin\chi \cos\psi \\ \eta &\approx (\tfrac{1}{2})[r(\tau + \sigma) + (\tau - \sigma)h \sin\chi \sin\psi] \\ z &\approx (\tfrac{1}{2})(\tau - \sigma)h \cos\chi \end{aligned}\right\} \quad (82)$$

Let the z coordinate of the blade mean surface be given by

$$z_b = -f(\xi, \psi) + \eta \tan[\theta(\xi, \psi)] \quad (83)$$

where $\theta(\xi, \psi)$ is the blade section pitch angle. The z coordinate of the reference surface in the vicinity of the ξ axis is given by[6]

$$z_r = \eta \tan\theta_0 \quad (84)$$

where

$$\tan\theta_0 = h \cos\chi/(r + h \sin\chi \sin\psi) \quad (85)$$

Then n_b is given by

$$n_b|_{\sigma=0} = (\mathbf{n}\cdot\mathbf{k})_{\sigma=0}(z_b - z_r) \quad (86)$$

Now θ can be written as

$$\theta = \theta_0 + \theta_1 \quad (87)$$

where θ_1 is considered small from the assumption of small disturbance. By using Eqs. (83–85, and 87), and by neglecting terms of higher orders in τ, Eq. (86) becomes

$$n_b|_{\sigma=0} = -\frac{r + h \sin\chi \sin\psi}{2H_0} f(r, \psi) + \tau \left\{ H_0\theta_1(r, \psi) + \frac{h \sin\chi \cos\psi}{4H_0} \left[\left(\frac{h \cos\chi}{2H_0}\right)^2 f(r, \psi) - (r + h \sin\chi \sin\psi) \frac{\partial f(r, \psi)}{\partial r} \right] \right\} \quad (88)$$

Then, from Eq. (25), there follows

$$\begin{aligned} w_n(r, \psi) &= w_n|_{\sigma=\tau=0} \\ &= \Omega \left\{ 2H_0\theta_1(r, \psi) - \frac{r + h \sin\chi \sin\psi}{2H_0} \times \left[\frac{\partial f(r, \psi)}{\partial \psi} + (h \sin\chi \cos\psi) \frac{\partial f(r, \psi)}{\partial r} \right] \right\} \end{aligned} \quad (89)$$

where

$$\theta_1(r, \psi) = \theta(r, \psi) - \tan^{-1}[h \cos\chi/(r + h \sin\chi \sin\psi)] \quad (90)$$

As for the blade section lift, the following approximation may be possible[6]:

$$L(r, \psi) \approx -H_0 \int_{\tau_{10}}^{\tau_{20}} \Delta p d\tau \quad (91)$$

By introducing Eq. (22) into the preceding equation, there follows:

$$L = \rho\Omega H_0[2\Gamma_0 + \tau_{20}(\partial\Gamma_0/\partial\psi) - \partial\Delta/\partial\psi] \quad (92)$$

According to Eq. (80), τ_{20} in the preceding equation is equal to b, which is given by Eq. (79). When Γ_0 and Δ are obtained by solving Eqs. (73) and (76), the section lift can be determined from Eq. (92).

Method for Solution

The equation of the circulation, Eq. (73), is very complicated mainly in that it involves the summation with respect to p in addition to the spanwise integration. By applying appropriate collocation techniques, the equation can be reduced to simultaneous linear algebraic equations as in the case of the lifting-line equation for the ordinary wing. However, the number of unknowns is much larger in this case than in the ordinary wing case because of the involvement of the p summation. The equation of the circulation moment also involves the summation by p, but this does not cause any complication since only the known function is summed.

The spanwise integrals in Eqs. (73) and (76) are generally of the form

$$\int_{r_1}^{r_2} K(r, r')\Omega(r')dr' \quad (93)$$

where $\Omega(r')$ represents $\partial\Gamma_0/\partial r'$, $G_p(r')$, $\partial G_p/\partial r'$, or $\partial\Delta/\partial r'$; and $K(r, r')$ is regarded as the kernel of the generalized integral equation. When $r' = r$, some of the kernels become infinitely large, as any one of the three kinds of singularities: $1/(r' - r)$, $\log|r' - r|$, and $[(r' - r)/|r' - r|] \log|r' - r|$, or as a combination of them.[7] The integration of Eq. (93) is carried out by dividing the kernel $K(r, r')$ into a singular part $K_S(r, r')$ and a nonsingular part $K_{NS}(r, r')$. Here K_S has the aforementioned singularities or, in addition, a finite discontinuity at $r' = r$, whereas K_{NS} is continuous throughout the blade span.

Change the variable from r to ϕ by the transformation

$$r = \tfrac{1}{2}[r_1 + r_2 + (r_2 - r_1) \cos\phi] \quad (94)$$

In the integrals involving K_S, apply the Multhopp interpolation[7,8]

$$\begin{aligned} f(\phi) &= \left(\frac{2}{n+1}\right) \sum_{i=1}^{n} f(\phi_i) \sum_{j=1}^{n} \sin(j\phi_i)\sin(j\phi) \\ \phi_i &= i\pi/(n+1) \end{aligned} \quad (95)$$

to Γ_0, G_p, or Δ, whereas in the integrals involving K_{NS}, in addition to the same manipulation apply the Tchebysheff interpolation[7,9]

$$g(\phi) = \left(\frac{2}{m+1}\right) \sum_{j=1}^{m+1} g(\phi_j) \left[\frac{1}{2} + \sum_{k=1}^{m} \cos(k\phi_j)\cos(k\phi)\right] \quad (96)$$

$$\phi_j = (2j - 1)\pi/2(m + 1)$$

to K_{NS}. Then all the spanwise integrals are reduced to linear combinations of the values of Γ_0, G_p, or Δ at the representative points having coordinates ϕ_i.

The summations with respect to p in Eqs. (73) and (76) can be written in the form

$$\sum_{p=-\infty}^{\infty} F(p)G_p(\phi_i)e^{ip\psi} \quad (97)$$

If only a finite number of terms of this series are retained and if G_p, the complex Fourier coefficient of Γ_0, is approximated by[10]

$$G_p(\phi_i) = \left(\frac{1}{2Q}\right) \sum_{q=0}^{2Q-1} \Gamma_0(\phi_i, \psi_q)e^{-ip\psi_q} \qquad \psi_q = \frac{q\pi}{Q} \quad (98)$$

instead of the rigorous integral expression, then the summations reduce to linear combinations of the values of Γ_0 at the representative azimuth angles ψ_q.

Through the foregoing manipulations, the equation of the circulation, Eq. (73), results in linear simultaneous algebraic equations with as many unknowns as the product of the numbers of the representative coordinates and the representative azimuth angles, whereas the equation of the circulation moment, Eq. (76), results in those with as many unknowns as the number of the representative coordinates for each representative azimuth angle.

The integrals with respect to λ from 0 to ∞ involved in some of the kernels can be evaluated numerically by using, for example, Filon's method.[11] The values of $A(k_p)$ and $B(k_p)$ are shown in Fig. 2 only for positive values of k_p. For negative values of k_p, $A(k_p)$ and $B(k_p)$ are respectively equal to the complex conjugates of $A(-k_p)$ and $B(-k_p)$.

Numerical Example

An example calculation was carried out for the case of an experiment of Ref. 12 with the tip speed ratio 0.2. Since the ratio of the blade root radius to the tip radius, r_1/r_2, is equal to 0.205 for this case, the blade always runs outside the reversed flow region. Tip-path plane was taken as the xy plane defined in this theory. The z component of the uniform flow velocity, W, was considered to be the sum of the component of the flight speed and the mean induced velocity at the disk except in the tangential flow condition where the latter was excluded. The mean induced velocity was determined from the experimental value of the thrust by the use of Glauert's formula.

Five representative coordinates and ten representative azimuth angles were used. Therefore, the final form of the equation of the circulation is a system of 50 linear simultaneous equations whereas that of the equation of the circulation moment consists of ten systems of five simultaneous equations.

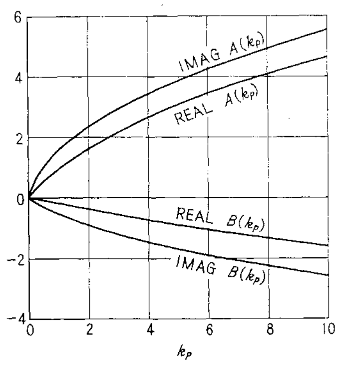

Fig. 2 $A(k_p)$ and $B(k_p)$ vs k_p.

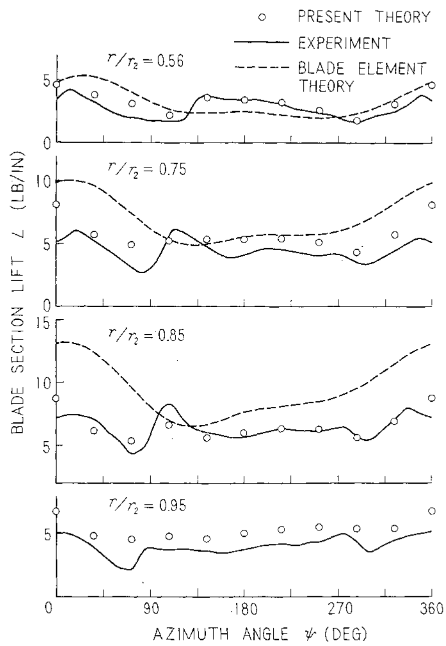

Fig. 3 Comparisons of theoretical and experimental lift variations (experimental results from Ref. 12, tip speed ratio 0.2).

Azimuthwise lift variations at several blade sections are shown in Fig. 3. The values obtained by the present theory have been cross plotted from the radial lift distribution at each representative azimuth angle in order to show the values at blade sections chosen in the experiment. In the calculation using blade element theory, the blade angle of attack was corrected with the same mean induced velocity as stated previously.

As compared with the experimental values, the present theory is seen generally to show a fairly good agreement except for a few unsatisfactory points. The first of these points is that the theory gives, in general, unconservative values at $r/r_2 = 0.75$ and 0.95. The second is that the theory does not seem to be able to predict satisfactorily an abrupt change experienced in the range of $\psi = 70°–120°$. The third is that a large difference is seen at $\psi = 0°$.

As regards the first point, it is difficult to give clear explanations. Since accuracy of the experiment has not necessarily been ascertained, it may be possible that the blade has been deformed elastically, which has not been incorporated in the calculation of course. It is probable from a simplified calculation that the theoretical lift at $r/r_2 = 0.95$ moves on towards the experimental value when the number of the representative coordinates is increased.[7]

As pointed out by Scheiman,[13] the abrupt lift change of the second point can be explained to be caused by the tip vortex of the preceding blade. Since the tip vortex is considered to be the result of the fact that the vortex sheet, trailed behind the blade tip region, has rolled up, and resembles a vortex tube, it may be possible that only somewhat gradual changes are obtained by the use of the linear theory in which the vortex sheet is assumed to stay on the reference surface. As for the position of the tip vortex, it not only comes somewhat inside the actual tip path through the rolling-up process and by the effect of contraction of the slip stream, but also exhibits a complicated variation in the vertical direction due to induced flow. These effects also, being outside the scope of the linear theory, may be one of the causes of the error.

The third error may probably be caused by the fact that the blade may enter the wake of the rotor hub or the mast at

$\psi = 0°$, and therefore may not be accounted for by the theory.

The number of representative coordinates and azimuth angles chosen in this example is not, of course, large enough, but the computer capacity did not permit the use of a greater number of them.

Conclusions

In this theory, the lifting-line equations of the rotor blade have been derived, through approximations as reasonable as possible, from the lifting-surface equation, which is exact within the scope of the linear theory. However, as seen from the numerical example, the effect of the distortion, especially the rolling-up, of the wake vortex sheet seems to be significant. If there exists a limitation of validity of the linear theory, this will be revealed through further comparisons between calculations by this theory and accurate experimental results.

When considered from a standpoint of the balance of the degree of approximations, this theory may possibly have unnecessarily precise portions as compared to the assumption neglecting the aforementioned nonlinear effects. It seems that further developments of the rotor theory should be directed to simplifying such portions, if they exist, and taking account of the distortion of the wake vortex sheet.

References

[1] Willmer, M. A. P., "The loading of helicopter rotor blades in forward flight," Aeronautical Research Council R&M 3318 (1959).

[2] Miller, R. H., "Rotor blade harmonic air loading," AIAA J. **2**, 1254–1269 (1964).

[3] Reissner, E., "Effect of finite span on the airload distributions for oscillating wings, I—Aerodynamic theory of oscillating wings of finite span," NACA TN 1194 (1947).

[4] Weissinger, J., "The lift distribution of swept-back wings," NACA TM 1120 (1947).

[5] Robinson, A. and Laurmann, J. A., *Wing Theory* (University Press, Cambridge, England, 1956), Chap. I, p. 28.

[6] Ichikawa, T., "Linearized aerodynamic theory of rotor blades (II), lifting-line theory," National Aerospace Lab., Tokyo, TR-85 (1965).

[7] Ichikawa, T., "Linearized aerodynamic theory of rotor blades (III), method for solving lifting-line equations," National Aerospace Lab., Tokyo, TR-100 (1966).

[8] Multhopp, H., "Die Berechnung der Auftriebsverteilung von Tragfluegeln," Luftfahrtforschung **15**, 153–169 (1938).

[9] Kopal, Z., *Numerical Analysis* (Chapman & Hall Ltd., London, 1955), App. II, p. 501.

[10] Lanzos, C., *Applied Analysis* (Prentice Hall Inc., Englewood Cliffs, N. J., 1956), Chap. IV, p. 229.

[11] Hamming, R. W., *Numerical Methods for Scientists and Engineers* (McGraw-Hill Book Company Inc., New York, 1962), Chap. XXIV, p. 319.

[12] Rabbott, J. P., Jr. and Churchill, G. B., "Experimental investigation of the aerodynamic loading of a helicopter rotor blade in forward flight," NACA RM L56I07 (1956).

[13] Scheiman, J. and Ludi, L. H., "Qualitative evaluation of effect of helicopter rotor-blade tip vortex on blade airloads," NASA TN D-1637 (1963).

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Structures Cost Effectiveness

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The principles of structures cost effectiveness are discussed, and an analytical method is presented to establish the optimum structural material/design concept that provides the best balance between weight and cost for a given vehicle. The criterion for selection of the optimum cost-weight design is dependent upon the value of weight saving applied to the vehicle under study. This value, or worth of saving weight, is derived in a sample analysis of the total systems costs associated with a fighter aircraft representative of typical current vehicles. It is shown that the optimum design for structures cost effectiveness may differ from the minimum-weight concept, and that the merit function used in this method may be applied to any structural design problem wherein weight and cost are the prominent evaluation factors. In several illustrative examples, the structures cost effectiveness approach is applied to various design areas in a typical high-aspect-ratio wing box.

Nomenclature

C_b = basic material cost, \$/lb
C_i = cost of producing ith design concept
E = modulus of elasticity, psi
L = column length, in.
N_x = distributed axial load, lb/in.
$\bar{t}$ = effective thickness, in.
V, V' = vehicle tradeoff value or slope, and component cost-weight slope, respectively
W_i = structural weight of ith design concept
$\Delta C, \Delta C'$ = change in cost for vehicle and structural components, respectively
$\Delta W, \Delta W'$ = change in weight for vehicle and structural components, respectively
Φ = cost-weight merit function
η = plasticity correction factor
ϵ = efficiency factor
ρ_m = material density, lb/in.^{3}

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Introduction

THE ever-increasing demand for lightweight aerospace hardware, coupled with the rising costs of producing efficient structural designs, has given impetus to an effort to relate weight and costs in a rational manner. Absolute minimum-weight designs are frequently impractical to assemble, and even when producible, are often extremely costly